Number-Theoretic Algorithms (RSA and related algorithms)

Chapter 31, CLRS book

Outline

- Modular arithmetic
- RSA encryption scheme
- Miller-Rabin algorithm (a probabilistic algorithm)

Modular Arithmetic

Integers

- $a \mid b$: a divides b, a is a divisor of b.
- $\gcd(a, b)$: greatest common divisor of a and b.
- Coprime or relatively prime: $\gcd(a, b) = 1$.
- Euclid's algorithm: compute $\gcd(a, b)$.
- Extented Euclid's algorithm: compute integers x and y such that $ax + by = \gcd(a, b)$.

Integers modulo n

- Let $n \geq 2$ be an integer.
- Definition: a is congruent to b modulo n, written $a \equiv b \mod n$, if $n \mid (a-b)$, i.e., a and b have the same remainder when divided by n.
- Note: $a \equiv b \bmod n$ and $a = b \bmod n$ are different.
- Definition: $[a]_n = \{x \in Z : x \equiv a \bmod n\}$.
- $[a]_n$ is called a residue class modulo n, and a is a representative of that class.

- There are exactly n residue classes modulo n:

  $[0],\ [1],\ [2],\ \ldots,\ [n-1].$

- If $x \in [a],\ y \in [b],$ then $x+y \in [a+b]$ and $x \cdot y \in [a \cdot b].$

- Define addition and multiplication for residue classes:

$$[a] +_n [b] = [a+b]$$

$$[a] \cdot_n [b] = [a \cdot b].$$

Group

- A group, denoted by $(G,*)$, is a set G with a binary operation $*$ such that

  1. $\forall x, y \in G,\ x * y \in G$ (closure)
  1. $x*(y*z)=(x*y)*z$ (associativity)
  2. $\exists\ e \in G$ s.t. $\forall x \in G,\ e*x = x*e = x$ (identity)
  3. $\forall x \in G,\ \exists\ y \in G$ s.t. $x*y = y*x = e$ (inverse)

- A group $(G,*)$ is abelian if $\forall x, y \in G,\ x*y = y*x$.

- Examples: $(Z,+)$, $(Q,+)$, $(Q\setminus\{0\},\ \times)$, $(R,+)$, $(R\setminus\{0\},\ \times)$.

- Define $Z_n = \{[0],\ [1],\ \ldots,\ [n-1]\}$.
- Or, more conveniently, $Z_n = \{0,\ 1,\ \ldots,\ n-1\}$.
- $(Z_n, +)$ forms an abelian additive group.
- For $a, b \in Z_n$,
  - $a + b = (a+b) \bmod n$. (Or, $[a]+[b]=[a+b]=[a+b \bmod n]$.)
  - 0 is the identity element.
  - The inverse of a, denoted by $-a$, is $n-a$.
- When doing addition/substraction in Z_n, just do the regular addition/substraction and reduce the result modulo n.
  - In Z_{10}, $5+5+9+4+6+2+8+3=?$

- $(Z_n, *)$ is not a group, because 0^{-1} does not exist.
- Even if we exclude 0 and consider only $Z_n^+ = Z_n \setminus \{0\}$,

  $(Z_n^+, *)$ is not necessarily a group; some a^{-1} may not exist.
- For $a \in Z_n$, a^{-1} exists if and only if $\gcd(a, n) = 1$.

- Let $Z_n^* = \{a \in Z_n : \gcd(a,n) = 1\}$.
- $(Z_n, *)$ is an abelian multiplicative group.
- $a * b = ab \mod n.$
  - $a * b = ab \mod n.$
  - 1 is the identity element.
  - The inverse of a, written a^{-1}, can be computed by the Extended Euclidean Algorithm.
- For example, $Z_{12}^* = \{1, 5, 7, 11\}$. $5 * 7 = 35 \bmod 12 = 11.$
- Q: How many elements are there in Z_n^*?

- Euler's totient function:

$$\varphi(n) = \left| Z_n^* \right|$$

$$= \left| \{ a : \ a \in Z_n \ \text{and} \ \gcd(a,n) = 1 \} \right|$$

- Facts:

1. $\varphi(p^e) = (p-1)\,p^{e-1}$ for prime p
2. $\varphi(ab) = \varphi(a)\,\varphi(b)$ if $\gcd(a,b) = 1$

- Let G be a (multiplicative) finite group.
- Lagrange's theorem: For any element $a \in G$, $a^{|G|} = e$.
- Corollary: For any element $a \in G$, $a^m = a^{m \bmod |G|}$.

- Euler's theorem:

  If $a \in Z_n^*$ (for any $n > 1$), then $a^{\varphi(n)} = 1$ in Z_n^*.

- Fermat's little theorem:

  If $a \in Z_p^*$ (p a prime), then $a^{\varphi(p)} = a^{p-1} = 1$ in Z_p^*.

Example: $n = 15$

- $Z_{15}^{*} = \{1, 2, 4, 7, 8, 11, 13, 14\}$
- $\left|Z_{15}^{*}\right| = \varphi(15) = \varphi(3) \times \varphi(5) = 2 \times 4 = 8$
-

$a \in Z_{15}^{*}$:	1	2	4	7	8	11	13	14
$\mathrm{ord}(a)$:	1	4	2	4	4	2	4	2

- $\mathrm{ord}(a)$: smallest integer k such that $a^k = 1$.
- $a^{\varphi(n)} = a^8 = 1$
- $13^{816243240481} = ?$

Algorithms

- $\gcd(a,b)$
- $a^{-1} \bmod n$
- $a^k \bmod n$
- Running time: $O\left(\log^3 n\right)$
- Here we assume $a,b \in Z_n$.

Euclid's Algorithm

- Given $n > a > b \geq 0$, compute $\gcd(a,b)$. $(a, b \in Z_n)$
- Theorem: If $b = 0$, $\gcd(a,b) = a$.

  If $b > 0$, $\gcd(a,b) = \gcd(b, a \bmod b)$
- $\text{Euclid}(a,b)$

  if $b = 0$

  then $\text{return}(a)$

  else $\text{return}\left(\text{Euclid}(b, a \bmod b)\right)$
- The number of recursive calls to Euclid is $O(\log n)$.
- Computing $a \bmod b$ takes $O(\log^2 n)$.

Extended Euclidean Algorithm

Given $a > b \geq 0,$ compute x, y such that $d = \gcd(a,b) = ax + by.$

Example: $\gcd(299, 221) = ?$

$$299 = 1 \times 221 + 78$$

$$221 = 2 \times 78 + 65$$

$$78 = 1 \times 65 + 13$$

$$65 = 5 \cdot 13 + 0$$

$$\begin{aligned}\gcd(229, 221) &= 13 = 78 - 65 \\ &= 78 - (221 - 2 \times 78) = 3 \cdot 78 - 221 \\ &= 3 \times (299 - 1 \cdot 221) - 221 \\ &= 3 \times 299 - 4 \times 221\end{aligned}$$

Extended Euclidean Algorithm

Given $a > b \geq 0$, compute d, x, y such that $\gcd(a,b) = d = ax + by$.

$\text{Extended-Euclid}(a,b)$

if $b = 0$ then

$\text{return}(a,1,0)$

else

$(d', x', y') \leftarrow \left(\text{Extended-Euclid}(b, a \bmod b)\right)$

$(d, x, y) \leftarrow \left(d', y', x' - \lfloor a/b \rfloor y'\right)$

$\text{return}(d, x, y)$

Correctness Proof

- If $b = 0$, $\gcd(a,b) = a = a \cdot 1 + b \cdot 0$.

  The returned answer $(a,1,0)$ is correct.

- If (d', x', y') is correct,

  $\Rightarrow \gcd(b,\ a \bmod b) = d' = b \cdot x' + (a \bmod b) \cdot y'$

  $\Rightarrow \gcd(b,\ a \bmod b) = d' = b \cdot x' + \left(a - \lfloor a/b \rfloor \cdot b\right) \cdot y'$

  $\Rightarrow \gcd(a,\ b) = d' = a \cdot y' + b \cdot \left(x' - \lfloor a/b \rfloor y'\right)$

  $\Rightarrow (d,\ x,\ y) \leftarrow \left(d',\ y',\ x' - \lfloor a/b \rfloor y'\right)$ is correct

How to compute $a^{-1} \bmod n$?

- Compute a^{-1} in Z_n^*.
- a^{-1} exists if and only if $\gcd(a,n)=1$.
- Use extended Euclidean algorithm to find x, y such that $ax+ny=\gcd(a,n)=1 \quad (\text{in } Z)$

$$\begin{aligned}
&\Rightarrow [a][x]+[n][y]=[1] \\
&\Rightarrow [a][x]=[1] \qquad (\text{since } [n]=[0]) \\
&\Rightarrow [a]^{-1}=[x].
\end{aligned}$$

- Note: may omit [ ], but reduce everything modulo n.

Example

- Compute $15^{-1} \mod 47$.
- Using extended Euclidean algorithm, we obtain

$$\gcd(15, 47) = 1 = 15 \times 22 - 47 \times 7$$

$$15^{-1} \mod 47 = 22$$

That is, $15^{-1} = 22$ in Z_{47}^{*}

Algorithm: Square-and-Multiply(x, c, n)

Comment: compute $x^c \bmod n$, where $c = c_k c_{k-1} \ldots c_0$ in binary.

$z \leftarrow 1$

for $i \leftarrow k$ downto 0 do

$\quad z \leftarrow z^2 \bmod n$

$\quad$ if $c_i = 1$ then $z \leftarrow (z \cdot x) \bmod n$

return (z)

Note: $$x^c = \begin{cases} \left(x^{\lfloor c/2 \rfloor}\right)^2 & \text{if c is even} \\ \left(x^{\lfloor c/2 \rfloor}\right)^2 \cdot x & \text{if } c \text{ is odd} \end{cases}$$

Example: $11^{23} \bmod 187$

$$23 = 10111_b$$

$$z \leftarrow 1$$

$$z \leftarrow z^2 \cdot 11 \bmod 187 = 11 \quad \text{(square and multiply)}$$

$$z \leftarrow z^2 \bmod 187 = 121 \quad \text{(square)}$$

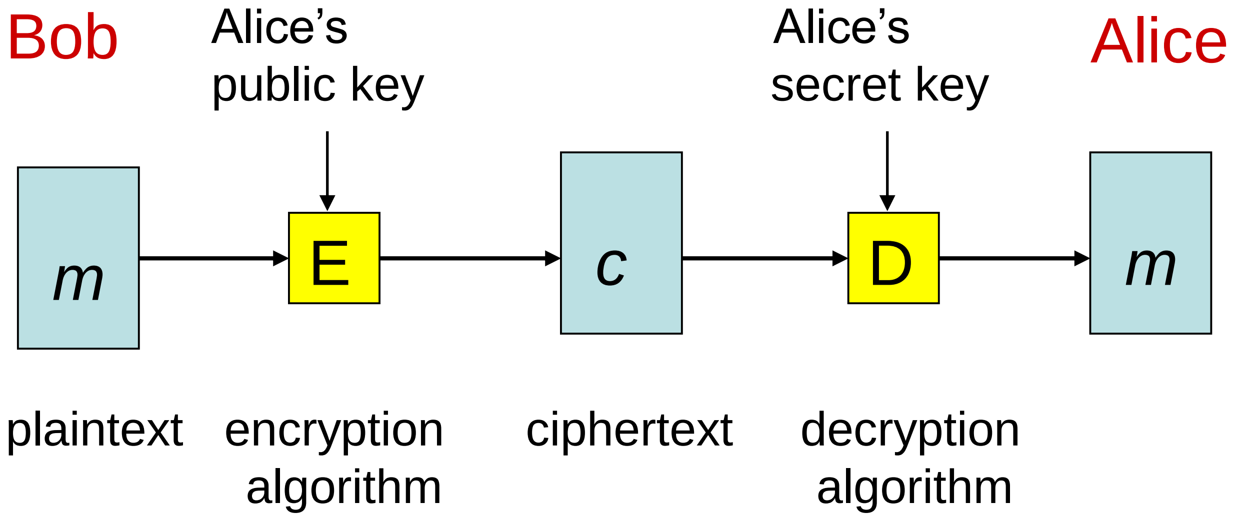
$$z \leftarrow z^2 \cdot 11 \bmod 187 = 44 \quad \text{(square and multiply)}$$

$$z \leftarrow z^2 \cdot 11 \bmod 187 = 165 \quad \text{(square and multiply)}$$

$$z \leftarrow z^2 \cdot 11 \bmod 187 = 88 \quad \text{(square and multiply)}$$

RSA Encryption

Public-key Encryption



The RSA Cryptosystem

- By Rivest, Shamir & Adleman of MIT in 1977.
- Best known and most widely used public-key scheme.
- Based on the assumed one-way property of modular powering:

$$f : x \to x^e \mod n \qquad \text{(easy)}$$

$$f^{-1} : x^e \to x \mod n \qquad \text{(hard)}$$

- In turn based on the hardness of integer factorization.

Idea behind RSA

It works in group Z_n^*. Let $x \in Z_n^*$ be a message.

Encryption (easy): $x \xrightarrow{\text{RSA}} x^e$

Decryption (hard): $x \xleftarrow{\text{RSA}^{-1}} x^e$

Decryption (easy with "trapdoor"): $x \xleftarrow{\text{RSA}^{-1}} x^e$

Looking for a "trapdoor": $(x^e)^d = x$.

If d is a number such that $ed \equiv 1 \bmod \varphi(n)$, then

$ed = k\varphi(n) + 1$ for some k, and

$$(x^e)^d = x^{ed} = x^{\varphi(n)k+1} = \left(x^{\varphi(n)}\right)^k \cdot x = 1 \cdot x = x.$$

RSA Cryptosystem

- Key generation:

  (a) Choose large primes p and q, and let $n := pq$.

  (b) Choose e $(1 < e < \varphi(n))$ coprime to $\varphi(n)$, and

  compute $d := e^{-1} \bmod \varphi(n)$. ($ed \equiv 1 \bmod \varphi(n)$.)

  (c) Public key: $pk = (n, e)$. Secret key: $sk = (n, d)$.

- Encryption: $E_{pk}(x) := x^e \bmod n$, where $x \in Z_n^*$.
- Decryption: $D_{sk}(y) := y^d \bmod n$, where $y \in Z_n^*$.

RSA Example: Key Setup

- Select two primes: $p = 17,\ q = 11.$
- Compute the modulus $n = pq = 187.$
- Compute $\varphi(n) = (p-1)(q-1) = 160.$
- Select e between 0 and 160 such that $\gcd(e, 160) = 1.$
  Say $e = 7.$
- Compute $d = e^{-1} \bmod \varphi(n) = 7^{-1} \bmod 160 = 23$
  (using extended Euclid's algorithm).
- Public key: $pk = (e,\ n) = (7,\ 187).$
- Secret key: $sk = (d,\ n) = (23,\ 187).$

RSA Example: Encryption & Decryption

- Suppose $m = 88.$
- Encryption: $c = m^e \bmod n = 88^7 \bmod 187 = 11.$
- Decryption: $m = c^d \bmod n = 11^{23} \bmod 187 = 88.$
- When computing $11^{23} \bmod 187,$ we do not first compute 11^{23} and then reduce it modulo 187.
- Rather, use square-and-multiply, and reduce intermediate results modulo 187 whenever they get bigger than 187.

Attacks on RSA

Attacks on RSA

- There are many attacks on RSA:
  - brute-force key search
  - mathematical attacks
  - timing attacks
  - chosen ciphertext attacks
- The most important one is integer factorization:

  If the adversary can factor n into pq. Then he can calculate $\varphi(n) = (p-1)(q-1)$ and the secret key $d = e^{-1} \bmod \varphi(n)$.

Integer Factorization

- A difficult problem.
- More and more efficient algorithms have been developed.
- In 1977, RSA challenged researchers to decode a ciphertext encrypted with a modulus *n* of 129 digits (428 bits). Prize: $100. RSA thought it would take quadrillion years to break the code using fastest algorithms and computers of that time. Solved in 1994.
- In 1991, RSA put forward more challenges (called RSA numbers), with prizes, to encourage research on factorization.

RSA Numbers

- Each RSA number is a semiprime. (A number is semiprime if it is the product of two primes.)
- There are two labeling schemes.
  - by the number of decimal digits:
    RSA-100, ..., RSA-500, RSA-617.
  - by the number of bits:
    RSA-576, 640, 704, 768, 896, 1024, 1536, 2048.

RSA Numbers which have been factored

- RSA-100 (332 bits), 1991, 7 MIPS-year, Quadratic Sieve.
- RSA-110 (365 bits), 1992, 75 MIPS-year, QS.
- RSA-120 (398 bits), 1993, 830 MIPS-year, QS.
- RSA-129 (428 bits), 1994, 5000 MIPS-year, QS.
- RSA-130 (431 bits), 1996, 1000 MIPS-year, GNFS.
- RSA-140 (465 bits), 1999, 2000 MIPS-year, GNFS.
- RSA-155 (512 bits), 1999, 8000 MIPS-year, GNFS.
- RSA-160 (530 bits), 2003, Lattice Sieve.
- RSA-576 (174 digits), 2003, Lattice Sieve.
- RSA-640 (193 digits), 2005, Lattice Sieve.
- RSA-200 (663 bits), 2005, Lattice Sieve.

RSA-200 =

27,997,833,911,221,327,870,829,467,638,

722,601,621,070,446,786,955,428,537,560,

009,929,326,128,400,107,609,345,671,052,

955,360,856,061,822,351,910,951,365,788,

637,105,954,482,006,576,775,098,580,557,

613,579,098,734,950,144,178,863,178,946,

295,187,237,869,221,823,983.

Remark

- In light of current factorization technologies, RSA recommends using an n of 1024-2048 bits.

Generating large primes

To set up an RSA cryptosystem,

we need two large primes p and q.

How to generate a large prime number?

- Generate a random odd number n of desired size.
- Test if n is prime.
- If not, discard it and try a different number.

Primality test: Is n a prime?

- Can it be solved in polynomial time?
- A long standing open problem until 2002.
- AKS(Agrawal, Kayal, Saxena): $O\left(\left(\log n\right)^{12+\varepsilon}\right)$.
  - Later improved by others to $O\left(\left(\log n\right)^{10.5}\right)$, and then to $O\left(\left(\log n\right)^{6+\varepsilon}\right)$.
- In practice, Miller-Rabin's probabilistic algorithm is still the most popular --- much faster, $O\left(\left(\log n\right)^{3}\right)$.

Miller-Rabin primality test: Is n a prime?

- Using some characteristic property of prime numbers:
  - n is prime $\Leftrightarrow$ $\forall a \in 2..\sqrt{n},\ a$ does not divide n.

- Miller-Rabin's idea: look for some property $P(a)$ s.t.
  - n is prime $\Rightarrow$ For all $a \in Z_n^*,\ P(a) = true$

    n not prime $\Rightarrow$ For at most a portion $1/k$ of elements

    $a \in Z_n^*,\ P(a) = true$

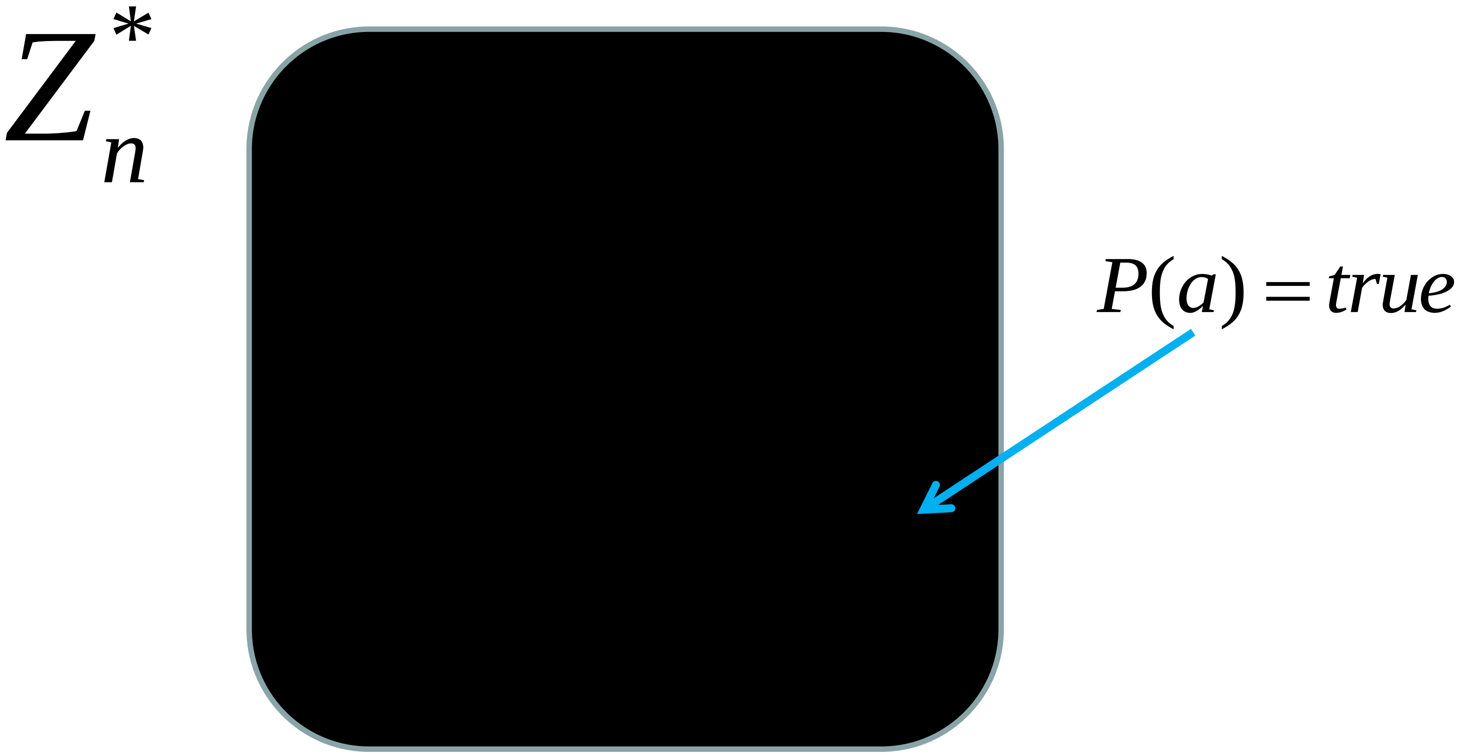
- Algorithm: Randomly pick t elements $a \in Z_n^*$.

  If $P(a)$ is true for all of them then return prime

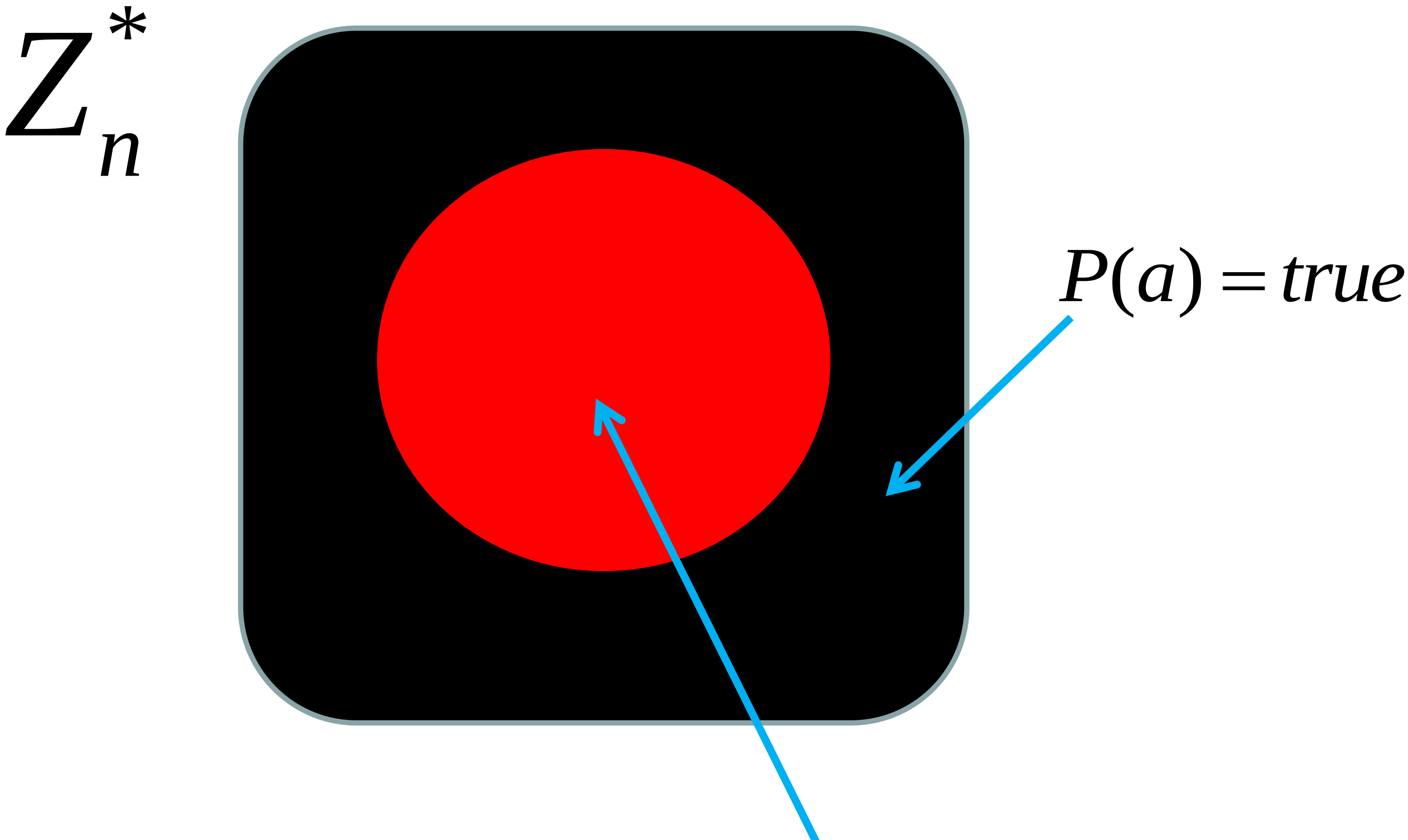
  else return composite.

- A "prime" answer may be incorrect with probability

  $\leq (1/k)^t$



If n is prime, then for all $a \in Z_n^*$, $P(a)$ is true.



If n is not prime, then there are strong witnesses,

which are elements $a \in Z_n^*$ s.t $P(a) = false$.

Say, at most $1/k$ of Z_n^* are black.

The property $P(a)$

- Write $n-1=u2^k$, where u is odd.
- Let $P(a)=\begin{cases} a^u \equiv 1 \mod n \text{ or} \\ a^{u2^i} \equiv -1 \mod n \text{ for some } i,\ 0 \le i \le k-1 \end{cases}$
- Consider the sequence

$$a^u,\ a^{u2},\ a^{u2^2},\ \ldots,\ a^{u2^{k-1}}$$

- If n is prime, then $P(a) = true$ for all $a \in Z_n^*$.
- If n is an odd composite and not a prime power, then at most one half of the elements $a \in Z_n^*$ are black (i.e., $P(a) = true$).
- A composite number n is a prime power if $n = p^e$ for some prime p and integer $e \geq 2$; a perfect power if $n = k^e$ for some integer k and $e \geq 2$.)

Algorithm: Miller-Rabin primality test

- Input: integer $n > 2$ and parameter t
- Output: a decision as to whether n is prime or composite

1. if n is even, return "composite"
2. if n is a perfect power, return "composite"
3. for $i := 1$ to t do

   choose a random integer a, $2 \le a \le n-1$

   if $\gcd(a, n) \ne 1$, return "composite"

   if a is a strong witness, return "composite"
4. return ("prime")

Analysis: Miller-Rabin primality test

- If the algorithm answers "composite", it is always correct.
- If the algorithm answers "prime", it may or may not be correct.
- The algorithm gives a wrong answer if n is composite but the algorithm fails to find a strong witness in t iterations.
- This may happen with probability at most 2^{-t}.
- Actually, at most 4^{-t}, by a more sophisticated analysis.

Monte Carlo algorithms

- A Monte Carlo algorithm is a probabilistic algorithm
  - which always gives an answer
  - but sometimes the answer may be incorrect.
- A Monte Carlo algorithm for a decision problem is yes-biased if its "yes" answer is always correct but a "no" answer may be incorrect with some error probability.
- A t-iteration Miller-Rabin is a "composite"-biased Monte Carlo algorithm with error probability at most $1/4^t$.

Las Vegas algorithms

- A Las Vegas algorithm is a probabilistic algorithm
  - which may sometimes fail to give an answer
  - but never gives an incorrect one
- A Las Vegas algorithm can be converted into a Monte Carlo algorithm.